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Constructing a completely integrable system via algebro-geometric data

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Abstract

We use the algebro-geometric data given by a genus-2 Jacobian, a curve and a line bundle on the Jacobian, and the action of a group of translates on the theta sections of this line bundle, to reconstruct an integrable system: the geodesic motion on $SO(4)$, metric II (so termed after Adler and van Moerbeke).

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1. Introduction

Since the early days of mechanics, finite-dimensional integrable systems have been related to algebraic geometry. This is shown in examples such as the rigid body cases or Jacobi's geodesic motion on the ellipsoid.

Most of the known examples are a particular class of integrable systems, whose solutions, expressible in terms of theta functions, are associated with Abelian varieties (i.e. complex tori in projective space) with divisors (codimension-one subvarieties) on them, and the Hamiltonian flows are linear on these Abelian varieties. Roughly speaking, such systems are called algebraic completely integrable (ACI).

Starting from an ACI system we can produce algebro-geometric data such as a divisor on an Abelian variety (the divisor at infinity), its polarization, the linear system associated with this divisor, and a finite group of translations leaving invariant the divisor and the holomorphic vector fields.

We can ask whether it is possible to go in the backward direction and view the integrable system as a deformation of suitable geometric data.

In this paper we show how to recover an algebraic completely integrable system from algebro-geometric data. The system considered is a geodesic motion on $SO(4)$ (the metric II case studied by Adler and van Moerbeke in [1, 2]). Here, the commuting complexified flows linearize on Jacobians A_α of genus-2 curves, upon adding to the complexified invariant manifolds, divisors D_α (curves called $SO(4)$ divisors) at infinity, with a precise pattern (they consist of four translates of the theta divisor intersecting at triple points, which are half-periods, like figure 1 below). One considers the action of the group of translations leaving invariant

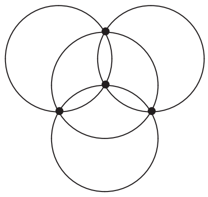

Figure 1.

D_α and the sections of the linear system of functions blowing up once at D_α and vanishing at least twice at the triple points. Surprisingly, this gives the sections for the right phase space. In the projective closure of the complexified phase space $\mathbb{C}^6$ (i.e. $\mathbb{P}^6$) the invariant manifolds compactify to set theoretical complete intersections, into which the Jacobians map birationally, so that a Jacobian minus its divisor at infinity is isomorphic to the respective (complexified) affine invariant manifold.

The question arises as to whether it is possible to reconstruct such a system by providing its Jacobian A_α, its configuration divisor at infinity $\mathcal{D}_\alpha$ (for instance an $SO(4)$ divisor as explained in theorem 1), a group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ of symmetries (which essentially are translates by half-periods in the Jacobi variety), and a line bundle $L_\alpha \to A_\alpha$, whose sections are projective coordinates of the ambient space $\mathbb{P}^6$. We provide such a construction by finding a convenient basis of theta functions for the above data, with the property that the same theta functions (up to permissible change of basis) are sections of the line bundle $L_\alpha \to A_\alpha$, and in which G has a 'nice' representation for all Jacobians. We deduce quadratic equations for the image of the Jacobians in $\mathbb{P}(H^0(A_\alpha, L_\alpha)^*) = \mathbb{P}^6$, in terms of certain parameters. Also, we find the quadratic equations for the holomorphic vector fields in terms of this basis.

The quadratic equations that describe the image of A_α in $\mathbb{P}^6$ contain natural parameters α which serve as the moduli data. Now, one of the theta sections, say Θ, will cut out on each A_α the divisor at infinity, and in the affine variables (Z_i/Θ) in $\mathbb{C}^6$ we obtain a smooth piece for each generic α (the affine piece). The question is whether such a family of affine surfaces put together in $\mathbb{C}^6$ has a Poisson structure so that they are the complexified invariant manifolds for a Hamiltonian structure. Indeed, such a Poisson structure, polynomial in the affine variables, is uniquely determined up to a Poisson transformation and a choice of Casimirs.

The above considerations lead us to the following theorem later.

Theorem 1. *Consider the family $\{A_\alpha\}$ of genus-2 Jacobians and divisors $\{\mathcal{D}_\alpha = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3\}$ on them, such that Θ_i is a translate by a half-period e_i of the theta divisor, and the Θ_is intersect into four triple points $\{e_0, \ldots, e_3\}$. This family possesses a group of translations $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ leaving invariant each $\mathcal{D}_\alpha$ and $A_\alpha \backslash \mathcal{D}_\alpha$. Let $H^0(A_\alpha, \mathcal{D}_\alpha - 2e_0 - 2e_1 - 2e_2 - 2e_3)$ be the space of sections linearly equivalent to $\mathcal{D}_\alpha$ that vanish at least twice at the e_is. Then, this space has dimension seven and decomposes into odd and even parts, with respect to the (-1)-involution, of dimensions one and six respectively, with the odd section vanishing at $\mathcal{D}_\alpha$.*

One can find a basis $\{v_i = Z_i/\Theta\}$, with Θ odd and the Z_is even, such that G acts as in table 1, for all generic Jacobians A_α:

Table 1.

	v_1	v_2	v_3	v_4	v_5	v_6
σ	$-v_2$	$-v_1$	$-v_3$	$-v_4$	$-v_6$	$-v_5$
τ	$-v_1$	$-v_2$	v_4	v_3	v_6	v_5

The image of A_α in $\mathbb{P}^6 = \mathbb{P}(H^0(A_\alpha, \mathcal{D}_\alpha - 2e_0 - 2e_1 - 2e_2 - 2e_3)^*)$ is a set theoretical complete intersection of four quadrics $q_1 = v_1v_2 = \alpha_1,\ q_2 = v_3v_4 = \alpha_2,\ q_3 = v_5v_6 = \alpha_3$ and another quadric $q_4 = \alpha_4$. If the quadrics q_2 and q_3 are chosen as Casimirs for a polynomial Poisson structure in the affine variables v_is, then, there is an integrable system with non-trivial Hamiltonians X_{q_1}, X_{q_4}, and a linear Poisson matrix. This system is, up to Poisson isomorphism, the metric II case of the geodesic motion on $SO(4)$ studied by Adler and Van Moerbeke [1, 9].

We use the normalized action of G and the tangency condition $X(q_i) = 0$ for any holomorphic vector field X and equation q_i to find all the equations of the image of the Jacobian in $\mathbb{P}^6$. The moduli parameters appear in the freedom we have to choose a different basis of sections for $L_\alpha \to A_\alpha$ so that the action of G on the equations of the variety and holomorphic vector fields are in a 'normal form' (i.e. roughly speaking, this means nice expressions without parameters).

The theta functions that would do the trick are products of genus-2 half-integer characteristic theta functions belonging to $H^0(A_\alpha, \mathcal{D}_\alpha - 2e_0 - 2e_1 - 2e_2 - 2e_3)$ (i.e. the system of theta functions whose zero locus is linearly equivalent to 4θ and vanish at least twice at the points e_0, e_1, e_2, e_3, which are half-periods and triple points of the configuration divisor at infinity $\mathcal{D}_\alpha$). Such a linear system was suggested in [3]. An explicit computation of the dimension of this space and higher powers of it did not come easily until a procedure by Bauer [4] was available. He considers the pull-back of linear systems on A to the surface A_Ω (the blow up of the Abelian surface A at the 16 half-periods). The computation can be done by using a bijection between symmetric curves on A_Ω and curves on the K3 surface $K_\Omega = A_\Omega/\{1, -1\}$ and then the Riemann–Roch formula and a theorem by Kodaira as explained in [10]. Some of the algebraic geometric assertions were already done by Szemberg [12] in his thesis. However, the step to reconstruct the integrable system is new and uses techniques already present in [9]. We are grateful to W Barth, Th Bauer and T Szemberg for showing us their approach to quadrics in $\mathbb{P}^6$ and to Pol Vanhaecke for useful discussions about this problem.

2. Preliminaries

Let $p : \tilde{A} \to A$ be the blow up of the Abelian surface A at the 16 half-periods $\{e_0, \ldots, e_{15}\}$, and $\{E_i = p^{-1}e_i, i = 0, \ldots, 15\}$ the 16 (-1)-curves. Let us denote by $(-1)_A$ the reflection with respect to the origin in A. This reflection induces an involution $(-1)_{\tilde{A}}$ in $\tilde{A}$. The quotient by the action of this involution $\tilde{K}_A = \tilde{A}/\langle(-1)_{\tilde{A}}\rangle$ is a smooth K3 surface and the projection $\pi : \tilde{A} \to \tilde{K}_A$ has the disjoint union of the 16 (-1)-curves E_i as a ramification divisor, and the disjoint union of the 16 (-2)-curves $B_i = \pi(E_i)$ as a branch locus.

Let $\mathcal{D}$ be a curve in A with multiplicities μ_i at the half-periods e_i and let ν_i be given non-negative integers for each i. We start from a symmetric divisor $\mathcal{D}$ (given by an even or odd section in $H^0(A, [\mathcal{D}])$) and consider the line bundle $\mathcal{L}_\nu$ on $\tilde{A}$ generated by $p^*(\mathcal{D}) - \sum \nu_i E_i = \widehat{\mathcal{D}} + \sum(\mu_i - \nu_i)E_i$ ($\widehat{\mathcal{D}}$ is the strict transform of $\mathcal{D}$). Then $\mathcal{L}_\nu$ is symmetric with respect to $(-1)_{\tilde{A}}$ (i.e. $\mathcal{L}_\nu \simeq (-1)^*_{\tilde{A}}\mathcal{L}_\nu$) if the ν_is have the same parity (proposition 3.1 [10]). The space $H^0(\tilde{A}, \mathcal{L}_\nu)$ is identified under p_* with the sections in $H^0(A, [\mathcal{D}])$ that vanish to order $\geqslant \nu_i$ at the e_is. Also, if $\mathcal{D}'$ and ν'_i are another divisor and positive integers, one has the intersection formula $(p^*(\mathcal{D}) - \sum \nu_i E_i) \cdot (p^*(\mathcal{D}') - \sum \nu'_i E_i) = \mathcal{D} \cdot \mathcal{D}' - \sum \nu_i \nu'_i$.

Let us consider $\mathcal{L}_\nu$ to be symmetric and let $(-1)_{\mathcal{L}_\nu}$ be the involution of $\mathcal{L}_\nu$ over $(-1)_{\tilde{A}}$ induced by the corresponding involution $(-1)_{[\mathcal{D}]}$ of $[\mathcal{D}]$ over $(-1)_A$. The action of $(-1)_{[\mathcal{D}]}$ on the fibres over the half-periods e_i is multiplication by $s_i = +1$ (in which case the half-period e_i is called even) or by $s_i = -1$ (where e_i is called odd).

There is an involution on sections $\varphi : H^0(\mathcal{L}_\nu) \to H^0(\mathcal{L}_\nu)$ defined by $\varphi(s) = (-1)_{\mathcal{L}_\nu} s(-1)_{\tilde{A}}$. This involution splits $H^0(\mathcal{L}_\nu)$ into (+1) and (-1) eigenspaces: $H^0(\mathcal{L}_\nu)^{\pm}$. Moreover, $\pi_*\mathcal{L}_\nu = \mathcal{M}^+ \oplus \mathcal{M}^-$ is a rank-2 bundle over $\tilde{K}_A$ which decomposes, with regard to $s \mapsto (-1)_{\mathcal{L}_\nu} s(-1)_{\tilde{A}}$, into (+1) and (-1) line bundles $\mathcal{M}^{\pm}$, and there are isomorphisms $H^0(\tilde{K}_A, \mathcal{M}^{\pm}) \simeq H^0(\tilde{A}, \mathcal{L}_\nu)^{\pm}$ [4].

Let $\widetilde{\mathcal{D}} = \widehat{\mathcal{D}} + \sum(\mu_i - \nu_i)E_i$ be a symmetric effective curve in the linear system $|\mathcal{L}_\nu|$, then, one can associate a curve in $\tilde{K}_A$ as follows: $\widetilde{\mathcal{C}} = \pi(\widehat{\mathcal{D}}) + \sum \left[\frac{\mu_i - \nu_i}{2}\right] B_i$, where the square brackets denote the integer part, $B_i = \pi_* E_i = \pi(E_i)$ and $\pi(\widehat{\mathcal{D}})$ is the image of $\widehat{\mathcal{D}}$. Starting from an odd or even curve $\mathcal{D}$ in the linear system $|\mathcal{D}|$ on A, we construct the curve $\widetilde{\mathcal{C}}$ on $\tilde{K}_A$ in this way associated with the divisor $p^*(\mathcal{D}) - \sum \nu_i E_i$. Then, by proposition 3.1 in [10] we have $\mathcal{M}^+ = \mathcal{O}_{\tilde{K}_A}(\widetilde{\mathcal{C}})$ if ν and $\mathcal{D}$ have the same parity, and $\mathcal{M}^- = \mathcal{O}_{\tilde{K}_A}(\widetilde{\mathcal{C}})$ otherwise.

The Riemann–Roch formula for a K3 surface $\tilde{K}_A$ and an effective curve $\mathcal{C}$ on it goes as follows:

$$h^0(\mathcal{C}) = \tfrac{1}{2}\mathcal{C}^2 + 2 + h^1(\mathcal{C}) \tag{1}$$

where $h^1(\mathcal{C}) = m - 1$ and m is the number of connected components of $\mathcal{C}$ (see [10]). This gives an effective method of computing the dimensions of $H^0(\mathcal{L}_\nu^{\otimes n})$ since we have an isomorphism $H^0(\mathcal{L}_\nu) \simeq H^0(\mathcal{M}^+) \oplus H^0(\mathcal{M}^-)$. Also, by Bauer [4], we have the formulae

$$\pi^*\mathcal{M}^{\pm} = \mathcal{L}_\nu \otimes [Z^{\mp}]^{-1} \tag{2}$$

where $Z^{\pm} = \sum_{s_i = \pm 1} E_i$ and s_i is the parity of the half-period e_i.

Example 1. *Let $\mathcal{D}$ be the divisor of figure 1 on a genus-2 Jacobian. This curve contains all 16 half-periods. The ones that are triple points are labelled $\{e_0, e_1, e_2, e_3\}$. $\mathcal{D}$ is an odd divisor in the linear system $|4\Theta|$ with respect to the (-1) involution that fixes the half-periods (lemma 7.7.1 [6]). Now, consider the bundle $\mathcal{L}_\nu = [p^*(\mathcal{D}) - \sum_{i=0}^3 2E_i]$ on $\tilde{A}$. We want to compute the dimensions of $H^0(\tilde{A}, \mathcal{L}_\nu)^{\pm}$.*

(a) Let us find $h^0(\mathcal{L}_\nu)^-$. We have that $p^*(\mathcal{D}) = \sum_{i=0}^3 D_i + \sum_{i=0}^3 3E_i + \sum_{i=4}^{15} E_i$, where the D_is represent the genus-2 curves. Then, the curve $\mathcal{D}^- = \sum_{i=0}^3 D_i + \sum_{i=0}^{15} E_i$ belongs to $|\mathcal{L}_\nu|^-$, and the associated curve in $\tilde{K}_A$ is $\mathcal{C}^- = \sum_{i=0}^3 \pi(D_i)$. One obtains $\pi^*\mathcal{C}^- = p^*\mathcal{D} - (\sum_{i=0}^3 3E_i + \sum_{i=4}^{15} E_i)$. Therefore, by formula (1),

$$h^0(\mathcal{L}_\nu)^- = h^0([\mathcal{C}^-]) = \tfrac{1}{4}(\pi^*\mathcal{C}^-)^2 + 2 + h^1(\mathcal{C}^-) = -4 + 2 + 3 = 1.$$

(b) Let $\mathcal{D}^+ \in |p * \mathcal{D} - \sum_{i=0}^3 2E_i|^+$. First, one constructs the divisors $Z^{\pm}$ as above. Taking into account that $\mathcal{D}$ is a totally symmetric divisor and that the parity of the origin is even, all periods turn out to be even. So, we obtain $Z^+ = \sum_{i=0}^{15} E_i$ and $Z^- = 0$. Let us denote by $B^{\pm}$ the direct image of $Z^{\pm}$, respectively. By applying π_* at the level of curves in formulae (2) we obtain the linear equivalence $2\mathcal{C}^+ + B^- \sim 2\mathcal{C}^* + B^+ - \sum_{i=0}^3 2B_i$, where $\mathcal{C}^+$ and $\mathcal{C}^*$ are the associated curves to $\mathcal{D}^+$ and $p^*\mathcal{D}$, respectively. This leads to the equivalence $2\mathcal{C}^+ \sim 2\sum_{i=0}^3 \pi(D_i) + \sum_{i=0}^{15} B_i$, or by pulling back to $\tilde{A}$: $\pi^*\mathcal{C}^+ \sim \sum_{i=0}^3 D_i + \sum_{i=0}^{15} E_i = p^*\mathcal{D} - \sum_{i=0}^3 2E_i$. Therefore, we calculate $(\mathcal{C}^+)^2 = \frac{1}{2}(\pi^*\mathcal{C}^+)^2 = 8$. It follows that such a curve on a K3 surface has $h^1(\mathcal{C}^+) = 0$ [11]. Then, by (1) $h^0(\mathcal{L}_\nu)^+ = h^0(\mathcal{C}^+) = 6$.

(c) From (a) and (b) we conclude that the space $H^0(\tilde{A}, \mathcal{L}_\nu)$ splits into a one-dimensional odd piece and a six-dimensional even part.

Example 2. *Compute the dimensions of the spaces $H^0(\mathcal{L}_\nu^{\otimes 2})^{\pm}$. We write $\pi_*\mathcal{L}_\nu^{\otimes 2} = \mathcal{M}_2^+ \oplus \mathcal{M}_2^-$ for the decomposition into ± 1 bundles of $\pi_*\mathcal{L}_\nu^{\otimes 2}$. By (a), in the above example, we have that $\pi^*\mathcal{M}^+ \simeq \mathcal{L}_\nu$. Therefore,*

$$\pi_*(\mathcal{L}_\nu^{\otimes 2}) \simeq \pi_*(\mathcal{L}_\nu \otimes \pi^*\mathcal{M}^+) \simeq \pi_*(\mathcal{L}_\nu) \otimes \mathcal{M}^+ \simeq (\mathcal{M}^+ \oplus \mathcal{M}^-) \otimes \mathcal{M}^+$$
$$\simeq (\mathcal{M}^+)^{\otimes 2} \oplus (\mathcal{M}^- \otimes \mathcal{M}^+).$$

It follows that $\mathcal{M}_2^+ \simeq (\mathcal{M}^+)^{\otimes 2} = [2\mathcal{C}^+]$, and $\mathcal{M}_2^- \simeq \mathcal{M}^- \otimes \mathcal{M}^+ = [\mathcal{C}^- + \mathcal{C}^+]$, because $(\mathcal{M}^+)^{\otimes 2}$ and $\mathcal{M}^- \otimes \mathcal{M}^+$ are eigenspaces under the action of the involution. The self-intersection numbers of the divisors representing $\mathcal{M}^{\pm}$ are greater than 8. So, in both cases $h^1(\mathcal{M}^{\pm}) = 0$ [11]. Thus, by the Riemann–Roch formula we obtain $h^0(\mathcal{M}_2^+) = 4\frac{(\mathcal{C}^+)^2}{2} + 2 = 18$ and $h^0(\mathcal{M}_2^-) = \frac{(\mathcal{C}^+)^2 + (\mathcal{C}^-)^2 + 2\mathcal{C}^+\cdot\mathcal{C}^-}{2} + 2 = 6$. In this case, the dimension of $H^0(\tilde{A}, \mathcal{L}_\nu^{\otimes 2})$ turns out to be $h^0(\mathcal{M}_2^+) + h^0(\mathcal{M}_2^-) = 18 + 6 = 24$.

3. Genus-2 theta functions

Let τ be the 2×2 Riemann matrix of a (generic and principally polarized) genus-2 Jacobian. A pair of real vectors (m, m^*) is associated unequivocally with the point $m^* + m\tau$ of $\mathbb{C}^2$.

For the pair of row vectors (m, m^*) (called characteristics) we define the classical theta functions [6, section 8.5] as (1) below, where $e(z) = \exp(2\pi i z)$, $z \in \mathbb{C}$. They have the properties (2), (3), (3′), (4).

(1) $\vartheta_{m,m^*}(\tau,\zeta) = \sum_{\psi\in\mathbb{Z}^n} e(\frac{1}{2}(\psi + m)\tau^t(\psi + m) + (\psi + m)^t(\zeta + m^*))$
(2) $\vartheta_{m,m^*}(\tau,-\zeta) = \vartheta_{-m,-m^*}(\tau,\zeta)$
(3) $\vartheta_{m+\psi,m^*+\psi^*}(\tau,\zeta) = e(m^t\psi^*)\vartheta_{m,m^*}(\tau,\zeta)$ for $\psi, \psi^* \in \mathbb{Z}^n$
(3′) $\vartheta_{m,m^*}(\tau,\zeta + u\tau + u^*) = e(-\frac{1}{2}u\tau^t u - u^t(\zeta + u^*))e(-u^t m^*)\vartheta_{m+u,m^*+u^*}(\tau,\zeta)$.

We also use the customary notation

$$\vartheta_{m,m^*}(\tau,\zeta) = \vartheta\begin{bmatrix} m \\ m^* \end{bmatrix}(\tau,\zeta)$$

and agree to represent the point $m^* + m\tau$ either by $\begin{bmatrix} m \\ m^* \end{bmatrix}$ or $\begin{Bmatrix} m \\ m^* \end{Bmatrix}$, when τ is fixed.

If $\begin{Bmatrix} m \\ m^* \end{Bmatrix} \in \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ is a half-period, then we have the formula [8, proposition 3.14, chapter II, p 167].

(4) $\vartheta_{m,m^*}(\tau,-\zeta) = e(2m^t m^*)\vartheta_{m,m^*}(\tau,\zeta) = e_*(m^* + m\tau)\vartheta_{m,m^*}(\tau,\zeta)$.

There are 2^{2g} half-periods on an Abelian variety of dimension g. We say that a half-period of characteristic $\begin{Bmatrix} m \\ m^* \end{Bmatrix}$ is odd (even) if the factor $e_*(\begin{bmatrix} m \\ m^* \end{bmatrix}_\tau)$ is negative (positive).

For a genus-2 Jacobian the even half-period characteristics are given by

$$e_{35} = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \end{Bmatrix} \quad e_{23} = \begin{Bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{Bmatrix} \quad e_{45} = \begin{Bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{Bmatrix} \quad e_{13} = \begin{Bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{Bmatrix}$$

$$e_{12} = \begin{Bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{Bmatrix} \quad e_{25} = \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{Bmatrix} \quad e_{14} = \begin{Bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{Bmatrix} \quad e_{15} = \begin{Bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{Bmatrix}$$

$$e_{24} = \begin{Bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{Bmatrix} \quad e_{34} = \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{Bmatrix},$$

while the odd characteristics are the following:

$$e_0 = \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{Bmatrix} \qquad e_1 = \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{Bmatrix} \qquad e_2 = \begin{Bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{Bmatrix} \qquad e_3 = \begin{Bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{Bmatrix}$$

$$e_4 = \begin{Bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{Bmatrix} \qquad e_5 = \begin{Bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{Bmatrix}.$$

It follows that the theta functions $\vartheta[e_i]$ are odd functions with respect to the involution, while the $\vartheta[e_{ij}]$s are even. One has the relations $\overline{e_i} + \overline{e_j} = \overline{e_{ij}} + \overline{e_0}$ and $\sum_{i=0}^{5} \overline{e_i} = 0$ on the Jacobian.

The odd half-periods are the Weierstrass points of the theta divisor $\overline{\{\zeta : \vartheta_{0,0}(\tau, \zeta) = 0\}} = \Gamma$ and Γ is also the genus-2 curve into its Jacobian.

4. The action of a group

Any $SO(4)$ divisor $\mathcal{D}$ can be constructed from one of the 16 symmetric curves $\{\Gamma + \overline{e_i}, \Gamma + \overline{e_{ij}}\}$ by acting on it with a particular group of translates $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, t_1, t_2, t_3 = t_1 + t_2\}$, in such a way that G fixes the triple points. Actually, the four triple points define translates by half-periods that coincide with G, once one of these points is chosen as the origin. It is easy to see that the same divisor $\mathcal{D}$ is obtained from a single symmetric curve, say Γ, by letting G act on it. Moreover, Γ contains three half-periods that together define an origin and G. Therefore, there are $80 = \binom{6}{3} \times 4$ ways of giving an origin and a group.

Let Γ be defined as in the previous section. Call $\Theta_3 = t_3(\Gamma)$, $\Theta_2 = t_2(\Gamma)$, $\Theta_1 = t_1(\Gamma)$, $\Theta_0 = \Gamma$, and $e0, e1, e2, e3$ the triple points of $\mathcal{D} = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3$. Then,

Proposition 2 (See [12]). *A basis for $H^0(\tilde{A}, [p^*(\mathcal{D}) - \sum_{i=0}^{3} 2E_i])$ is given by the odd section $s_0 s_1 s_2 s_3$ and the even sections $s_0^2 s_1^2$, $s_0^2 s_2^2$, $s_1^2 s_2^2$, $s_1^2 s_3^2$, $s_2^2 s_3^2$, where s_0, s_1, s_2, s_3 are theta functions vanishing on Θ_0, Θ_1, Θ_2, Θ_3 respectively.*

We pick three points e_0, e_1, e_2 in $\Gamma = \overline{\{\zeta : \vartheta(\tau, \zeta) = 0\}}$, e_0 as origin and consider the group G generated by $e_1 - e_0$, $e_2 - e_0$. This has an extra element $e_{12} - e_0$. We write

$$t_1 = e_1 - e_0 = \begin{Bmatrix} 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{Bmatrix} = e_{24} + \begin{Bmatrix} 0 & 0 \\ -1 & 0 \end{Bmatrix}$$

$$t_2 = e_2 - e_0 = \begin{Bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{Bmatrix} = e_{14} + \begin{Bmatrix} -1 & 0 \\ 0 & 0 \end{Bmatrix}$$

$$t_3 = e_{12} - e_0 = \begin{Bmatrix} -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \end{Bmatrix} = e_4 + \begin{Bmatrix} -1 & 0 \\ -1 & 0 \end{Bmatrix}$$

and consider the translates of the ϑ-divisor Γ by these elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. These translates are given by the sections

$$s_o = \vartheta[e_{35}](\tau, \zeta) = \vartheta(\tau, \zeta) \qquad s_1 = \vartheta[e_{24}](\tau, \zeta)$$

$$s_2 = \vartheta[e_{14}](\tau, \zeta) \qquad s_3 = \vartheta[e_4](\tau, \zeta).$$

Thus, the zero locus of $\Theta(\tau, \zeta) = \vartheta[e_{35}]\vartheta[e_{24}]\vartheta[e_{14}]\vartheta[e_4]$ gives a typical $SO(4)$ divisor. As Θ is the product of three even sections and one odd section, Θ is odd.

Table 2.

	$s_o = \vartheta[e_{35}](\tau, \zeta)$	$s_1 = \vartheta[e_{24}]$	$s_2 = \vartheta[e_{14}]$	$s_3 = \vartheta[e_4]$
t_1	$\vartheta[e_{35}](\tau, \zeta + e_1 - e_0) = \vartheta[e_{24}]$	$\vartheta[e_{35}]$	$-\vartheta[e_4]$	$\vartheta[e_{14}]$
t_2	$\vartheta[e_{35}](\tau, \zeta + e_2 - e_0) = f(\zeta)\vartheta[e_{14}]$	$f(\zeta)\mathrm{i}\vartheta[e_4]$	$f(\zeta)\vartheta[e_{35}]$	$f(\zeta)\mathrm{i}\vartheta[e_{24}]$
t_3	$\vartheta[e_{35}](\tau, \zeta + e_{12} - e_0) = -g(\zeta)\vartheta[e_4]$	$g(\zeta)\mathrm{i}\vartheta[e_{14}]$	$g(\zeta)\vartheta[e_{24}]$	$g(\zeta)\mathrm{i}\vartheta[e_{35}]$

Table 3.

	u_{01}	u_{02}	u_{03}	u_{12}	u_{13}	u_{23}	Θ
t_1	u_{01}	u_{13}	u_{12}	u_{03}	u_{02}	u_{23}	$-\Theta$
t_2	$-f^4u_{23}$	f^4u_{02}	$-f^4u_{12}$	$-f^4u_{03}$	f^4u_{13}	$-f^4u_{01}$	$-f^4\Theta$
t_3	$-g^4u_{23}$	g^4u_{13}	$-g^4u_{03}$	$-g^4u_{12}$	g^4u_{02}	$-g^4u_{01}$	$g^4\Theta$

We will construct table 2 with the action of t_x defined by $t_x\vartheta(\zeta) = \vartheta(\zeta + x)$. This action is associated with the Schrödinger representation of the theta group (see [6, 9] for these matters).

Let $u_{ij} = s_i^2 s_j^2,\ i < j$, be the even sections of $[p^*(\mathcal{D}) - \sum_{i=0}^{3} 2E_i]$, then G acts as shown in table 3.

This action is similar to that described in [12] for a different basis. We needed to go down to the classical theta functions to make the action on Θ explicit. If we put $v_3 = \frac{u_{01}}{\Theta}$, $v_4 = \frac{u_{23}}{\Theta}$, $v_1 = \frac{u_{02}}{\Theta}$, $v_2 = \frac{u_{13}}{\Theta}$, $v_5 = \frac{u_{03}}{\Theta}$, $v_6 = \frac{u_{12}}{\Theta}$, we obtain table 1 of the theorem, with $\sigma = t_1$ and $\tau = t_2$.

Since this action admits a separable rescaling, that is, a rescaling in each set of variables: $\{v_1, v_2\}$, $\{v_3, v_4\}$ and $\{v_5, v_6\}$, then, by the very definition of the v_is, we have relations $v_1v_2 = c_1$, $v_3v_4 = c_2$, $v_5v_6 = c_3$, where now, new v_is are substituted in place of the old rescaled ones. These equations are three independent integrals with three free parameters for the would-be system.

5. The quadratic vector fields invariant under the group

The above sections $\{u_{01}, u_{02}, u_{03}, u_{12}, u_{13}, u_{23}, \Theta\}$ of $H^0(p^*(\mathcal{D}) - 2E_0 - 2E_1 - 2E_2 - 2E_3)$, yield a rational map $A \to \tilde{A} \to \mathbb{P}^6$ (which is not defined at $e0$, $e1$, $e2$ and $e3$). The degree of the image is $(p^*\mathcal{D} - \sum 2E_i)^2 = \mathcal{D}^2 + \sum 4E_i^2 = 32 - 4 \times 4 = 16$.

Now, let $\mathcal{L}$ be the line bundle associated with the divisor $p^*\Theta - \sum_{i=0}^{3} 2E_i$. As mentioned before $h^0(\mathcal{L}) = 7$ and $h^0(\mathcal{L})^+ = 6$, $h^0(\mathcal{L})^- = 1$. The vector fields in the affine variables v_i are given by Wronskians of an even section with the odd section, and these define sections in $H^0(\mathcal{L}^{\otimes 2})^+$. This follows from the properties of Wronskians [9]: $W_Y : H^0(\mathcal{L})^+ \otimes H^0(\mathcal{L})^- \to H^0(\mathcal{L}^{\otimes 2})^+$. Thus, the vector fields can be written quadratically in terms of the even sections, which follows from the proposition:

Proposition 3. *The map $S^2H^0(\mathcal{L})^+ \to H^0(\mathcal{L}^{\otimes 2})^+$ is surjective, and $H^0(\mathcal{L})^+ \otimes H^0(\mathcal{L})^- \to H^0(\mathcal{L}^{\otimes 2})^-$ is an isomorphism.*

Proof. We have canonical isomorphisms $H^0(\tilde{A}, \mathcal{L})^\pm \cong H^0(\tilde{K}_A, \mathcal{M}^\pm)$ and $H^0(\tilde{A}, \mathcal{L}^2)^\pm \cong H^0(\tilde{K}_A, \mathcal{M}_2^\pm)$ and consider the conclusions of example 2.

The hypothesis of theorem 1 by Saint-Donat [11] is checked out in [12]. Therefore, we conclude that there is a surjective morphism $S^2H^0(\tilde{K}_A, \mathcal{M}^+) \to H^0(\tilde{K}_A, (\mathcal{M}^+)^{\otimes 2}) \cong H^0(\tilde{K}_A, \mathcal{M}_2^+) \cong H^0(\mathcal{L}^{\otimes 2})^+$ from which follows the first part of the proposition.

For the second part, note that the sections $u_{ij} \otimes \Theta, i < j$, of $H^0(\mathcal{L}^{\otimes 2})^-$ are linearly independent of A. $\square$

Remark 1. The map

$$\begin{aligned} S^2H^0(\mathcal{L}) &= S^2(H^0(\mathcal{L})^+ \oplus H^0(\mathcal{L})^-) \\ &= S^2H^0(\mathcal{L})^+ \oplus (H^0(\mathcal{L})^+ \otimes H^0(\mathcal{L})^-) \oplus S^2H^0(\mathcal{L})^- \to H^0(\mathcal{L}^{\otimes 2})^+ \oplus H^0(\mathcal{L}^{\otimes 2})^- \end{aligned}$$

is surjective, and this means there are $4 = s^2h^0(\mathcal{L}) - h^0(\mathcal{L}^{\otimes 2}) = 28 - 24$ quadratic equations defining the image of $\tilde{A}$ in $\mathbb{P}^6$.

Since the non-trivial holomorphic vector fields on the Jacobians have to be tangent to the affine variety defined by the quadrics $q_1 = v_1v_2 = c_1$, $q_2 = v_3v_4 = c_2$, $q_3 = v_5v_6 = c_3$, and invariant under the translations σ, τ, we obtain

$$\begin{aligned} \dot{v}_1 &= v_1(\alpha_3(v_3 + v_4) + \alpha_5(v_5 + v_6)) = v_1 f_1 \\ \dot{v}_2 &= -v_2(\alpha_3(v_3 + v_4) + \alpha_5(v_5 + v_6)) = -v_2 f_1 \\ \dot{v}_3 &= v_3(\beta_1(v_1 - v_2) + \beta_5(v_5 - v_6)) = v_3 f_2 \\ \dot{v}_4 &= -v_4(\beta_1(v_1 - v_2) + \beta_5(v_5 - v_6)) = -v_4 f_2 \\ \dot{v}_5 &= v_5(\gamma_1(v_1 + v_2) + \gamma_3(v_3 - v_4)) = v_5 f_3 \\ \dot{v}_6 &= -v_6(\gamma_1(v_1 + v_2) + \gamma_3(v_3 - v_4)) = -v_6 f_3 \end{aligned} \tag{3}$$

which gives several two-dimensional families of vector fields non-vanishing on each variable.

6. The extra quadratic invariant

We want to find the remaining invariant under the group G which is killed by the quadratic vector fields. Such an invariant must be of the form

$$\begin{aligned} q_4 = {} & \alpha(v_1^2 + v_2^2) + \beta(v_3^2 + v_4^2) + \gamma(v_5^2 + v_6^2) + \delta(v_1 + v_2)(v_3 - v_4) \\ & + \epsilon(v_1 - v_2)(v_5 - v_6) + \eta(v_3 + v_4)(v_5 + v_6). \end{aligned} \tag{4}$$

This has to satisfy the equation $\dot{q}_4 = 0$ under all vector fields $\dot{}$, which leads to the linear system

$$\begin{aligned} &2\alpha_3\alpha + \beta_1\delta = 0 && \alpha_3\delta + 2\beta_1\beta = 0 \\ &2\alpha_5\alpha + \gamma_1\epsilon = 0 && \alpha_5\delta + \beta_1\eta + \gamma_3\epsilon = 0 \\ &\alpha_3\epsilon + \beta_5\delta + \gamma_1\eta = 0 && 2\beta_5\beta + \gamma_3\eta = 0 \\ &\alpha_5\epsilon + 2\gamma_1\gamma = 0 && \beta_5\eta + 2\gamma_3\gamma = 0 \end{aligned}$$

with the following rank-5 matrix

$$\begin{bmatrix} 2\alpha_3 & 0 & 0 & \beta_1 & 0 & 0 \\ 2\alpha_5 & 0 & 0 & 0 & \gamma_1 & 0 \\ 0 & 0 & 0 & \beta_5 & \alpha_3 & \gamma_1 \\ 0 & 0 & 2\gamma_1 & 0 & \alpha_5 & 0 \\ 0 & 2\beta_1 & 0 & \alpha_3 & 0 & 0 \\ 0 & 0 & 0 & \alpha_5 & \gamma_3 & \beta_1 \\ 0 & 2\beta_5 & 0 & 0 & 0 & \gamma_3 \\ 0 & 0 & 2\gamma_3 & 0 & 0 & \beta_5 \end{bmatrix}. \tag{5}$$

Lemma 4. *Besides the invariants v_1v_2, v_3v_4, v_5v_6, an extra quadratic invariant of the form (4) killed by two independent quadratic vector fields exists, if it has one of the following coefficients* $(\alpha, \beta, \gamma, \delta, \epsilon, \eta)$*:*

(a) $(0, -\frac{a}{2}, -\frac{1}{2a}, 0, 0, 1)$
(b) $(-\frac{a}{2}, -\frac{1}{2a}, 0, 1, 0, 0)$
(c) $(-\frac{1}{2}ab, -\frac{1}{2}\frac{a}{b}, -\frac{1}{2}\frac{b}{a}, \pm a, b, 1)$
(d) $(-\frac{a}{2}, 0, -\frac{1}{2a}, 0, 1, 0)$.

Proof. The conditions for the existence of an extra quadratic invariant of the form (4) are those that make matrix (5) have rank 5. This is the vanishing of 28 6×6 minors. We do not write these cumbersome expressions and leave it for the reader to check the computations.

Next, we write a table with the solution of these equations and a basis of the kernel of (5) (that is, the coefficients of (4)) under the condition that there must be two linearly independent vector fields which do not vanish on the variables v_i. Also, this is left to the reader: first we have assumed $\alpha_5\beta_1\gamma_3 = \alpha_3\beta_5\gamma_1$, and then $\alpha_5\beta_1\gamma_3 = -\alpha_3\beta_5\gamma_1$. □

Remark 2. Any of the quadrics (a), (b), (d) together with q_1, q_2 and q_3 lead to a reducible affine variety. This is not desirable since we want the images of irreducible Jacobians. Therefore, we discard those cases.

Remark 3. Let us consider the set theoretical complete intersection CA in $\mathbb{P}^6$ determined by the quadrics $q_i = c_i, i = 1, \ldots, 4$. This surface contains the image of A in $\mathbb{P}^6$, call it $\overline{A}$. Moreover, the degree of CA is $16 = 2 \times 2 \times 2 \times 2$, the same as the degree of $\overline{A}$. Therefore, the rational map $\overline{A} \to CA$ induced by $A \to \mathbb{P}^6$ is generically one-to-one, and this means that CA coincide with $\overline{A}$ on an open set. The image of $A - \mathcal{D}$ in $\overline{A}$ is smooth and one-to-one because it corresponds to lifting from $\tilde{K}_A$ to $\tilde{A}$, away from the branch locus, the smooth embedding $\tilde{K}_A \to \mathbb{P}^5$ given by the even sections. Also, $A - \mathcal{D}$ maps onto the affine piece $\{v \in \mathbb{C}^6 : q_i(v) = c_i, i = 1, \ldots, 4\}$, and since this piece is smooth by construction and contains the image of $A - \mathcal{D}$ in $\mathbb{P}^6$, it coincides with it. Thus, $A - \mathcal{D}$ is isomorphic to the affine piece determined by the four quadrics.

7. The Poisson matrix

Assume the vector fields we are looking at to have the following special form with matrix J polynomial in the affine variables:

$$\dot{w}_j = X_H(w_j) = (J(w) \cdot \text{grad}\, H(w))_j = \sum_{j=1}^{6} J_{jk}(w) \cdot \frac{\partial H}{\partial w_k}. \tag{6}$$

We want to find a Poisson structure of the form $\{f, g\}(w) = \langle \text{grad}\, f(w), J(w) \cdot \text{grad}\, g(w)\rangle$ so that the Hamiltonian vector fields correspond to the holomorphic vector fields already found.

Let T_x be a translation $T_x : A \longrightarrow A$ on the Abelian variety A, then, we have the equivariance relation $\mathrm{d}T_x \cdot X_H = X_H \circ T_x$. The action of $T_x \in G$ on generating functions is linear $T_x^*(w_i) = \sum_{j=1}^{6} \lambda_{ij} w_j$. Then

$$X_H(y)\left(T_x^*(w_i)\right) = \sum \lambda_{ij} \sum \mathcal{J}_j(w) \frac{\partial H}{\partial w_k}.$$

But

$$X_H(y)\big(T_x^*(w_i)\big) = \mathrm{d}(w_i \circ T_x) \circ X(y) = \mathrm{d}w_i\, \mathrm{d}T_x X$$
$$= \mathrm{d}w_i \cdot X(T_x(y)) = X(T_x(y))(w_i) = T_x^*(X(y)(w_i))$$

where $X(y)(w_i)$ is a polynomial in the variables $w_1, \ldots, w_6$.

Now, for a globally defined polynomial H invariant under G, we have $T_x^* H(w_1, \ldots, w_6) = H(T_x^* w_1, \ldots, T_x^* w_6) = H(w_1, \ldots, w_6)$, and for the set of functions $T_x^*(w_i)$, the exchange of differentials $\mathrm{d}T_x^*(w_i) = \sum_{j=1}^{6} \lambda_{ij}\, \mathrm{d}w_j$ occurs. This induces on derivations the transformation formula

$$\frac{\partial}{\partial\left(T_x^* w_i\right)} = \sum_{j=1}^{6} \mu_{ij} \frac{\partial}{\partial w_j}$$

where $(\mu_{ij}) = ((\lambda_{ij})^t)^{-1}$. Moreover,

$$T_x^*\left(\frac{\partial H}{\partial w_i}\right) = T_x^*\left(\left(\frac{\partial}{\partial w_i}\right)(H)\right) = \left(\left(\frac{\partial}{\partial w_i}\right) \circ T_x\right)(H \circ T_x) = \frac{\partial}{\partial(T_x^* w_i)}(T_x^* H)$$
$$= \sum \mu_{ij} \frac{\partial}{\partial w_j} T_x^* H = \sum \mu_{ij} \frac{\partial}{\partial w_j} H.$$

Therefore, we obtain the relation

$$\sum \lambda_{ij} J_{jk}(w) \frac{\partial H}{\partial w_k} = \sum T_x^* J_{ik}(w) \cdot \mu_{kj} \frac{\partial H}{\partial w_j}.$$

So, for any integral invariant H

$$\sum_k \left(\sum_j \lambda_{ij} J_{jk}(w) - \sum_l T_x^* J_{il}(w) \cdot \mu_{lj}\right) \frac{\partial H}{\partial w_k} = 0. \qquad (7)$$

This means that the functions $\epsilon_{ik}(w) = \sum_j \lambda_{ij} J_{jk}(w) - J_{ij}(T_x^* w) \cdot \mu_{jk}$ are killed by the gradients $\operatorname{grad} H$, for H an (integral of the motion) invariant by G. Namely, the vectors $\epsilon_i(w) = (\epsilon_{i1}(w), \epsilon_{i2}(w), \ldots, \epsilon_{i6}(w))$ belong to the tangent space of the affine variety $\{w : q_i(w) = c_i, i = 1, \ldots, 4\}$. Thus, there must be a function H, linear combination of the non-trivial invariants H_1, H_2, such that $\epsilon_i(w) = J(w) \cdot \operatorname{grad} H$.

Assuming that there are no non-trivial Hamiltonians which are linear in the affine coordinates, we conclude that $\epsilon_i(w) = 0$ if the matrix J has linear entries.

Indeed, for a non-trivial Hamiltonian H, the polynomials in $w_1, \ldots, w_6$, $(J(w) \cdot \operatorname{grad} H)$, $i = 1, \ldots, 6$, have at least degree two. So, it follows:

Lemma 5. *If the matrix J has linear entries, then it is equivariant by the action of translations in G, and skew-equivariant by the action of (-1)-involution. Namely, if $\Lambda(\sigma)$ is the matrix of the translate σ, $J(\sigma w) = \Lambda(\sigma) J(w) \Lambda(\sigma)^t$, $\sigma \in G$, and if $\Lambda(\iota)$ is the matrix of the (-1)-involution ι, $J(\iota w) = -\Lambda(\iota) J(w) \Lambda(\iota)^t$.*

Let us assume that the Poisson matrix of this would-be system has linear entries. As the invariants are quadratic, we deduce that the matrix J satisfies the relations described by lemma 5.

Let us write a general linear 6×6 matrix in 2×2 blocks,

$$J = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}. \qquad (8)$$

Let I be the 2×2 identity matrix and

$$D = \left\{ \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\}$$

the 2×2 transposition. In matrix notation, the operators σ, τ and the (-1)-involution ι are written as follows:

$$\sigma = \begin{bmatrix} -D & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -D \end{bmatrix} \qquad \tau = \begin{bmatrix} -I & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{bmatrix} \qquad \iota = \begin{bmatrix} -I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix}.$$

The invariance property of J is described by the following two relations:

$$\sigma \cdot J = \begin{bmatrix} DA_{11}D & DA_{12} & DA_{13}D \\ A_{21}D & A_{22} & A_{23}D \\ DA_{31}D & DA_{32} & DA_{33}D \end{bmatrix}$$

$$\tau \cdot J = \begin{bmatrix} A_{11} & -A_{12}D & -A_{13}D \\ -DA_{21} & DA_{22}D & DA_{23}D \\ -DA_{31} & DA_{32}D & DA_{33}D \end{bmatrix}.$$

We also use the fact that J is skew symmetric. Thus, it is enough to solve the following equations:

$$\begin{cases} \sigma \cdot A_{11} = DA_{11}D \\ \tau \cdot A_{11} = A_{11} \end{cases} \quad \begin{cases} \sigma \cdot A_{22} = A_{22} \\ \tau \cdot A_{22} = DA_{22}D \end{cases} \quad \begin{cases} \sigma \cdot A_{33} = DA_{33}D \\ \tau \cdot A_{33} = DA_{33}D \end{cases}$$
$$\begin{cases} \sigma \cdot A_{12} = DA_{12} \\ \tau \cdot A_{12} = -A_{12}D \end{cases} \quad \begin{cases} \sigma \cdot A_{23} = A_{23} \\ \tau \cdot A_{23} = DA_{23}D \end{cases} \quad \begin{cases} \sigma \cdot A_{13} = DA_{13}D \\ \tau \cdot A_{13} = -A_{13}D. \end{cases} \tag{9}$$

The equations for $A_{11} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be written as $\begin{bmatrix} \sigma a & \sigma b \\ \sigma c & \sigma d \end{bmatrix} = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$ and $\begin{bmatrix} \tau a & \tau b \\ \tau c & \tau d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. This means that $A_{11} = \begin{bmatrix} a & b \\ \sigma a & \sigma b \end{bmatrix}$, with $a = \tau a, b = \tau b$.

Analogously, for the remaining A_{ij} we obtain $A_{22} = \begin{bmatrix} a & b \\ \tau b & \tau a \end{bmatrix}$, with $a = \sigma a, b = \sigma b$. $A_{33} = \begin{bmatrix} a & b \\ \sigma b & \sigma a \end{bmatrix}$ with $\tau\sigma b = b, \tau\sigma a = a$,

$$A_{13} = \begin{bmatrix} a' & -\tau a' \\ -\sigma\tau a' & \sigma a' \end{bmatrix} \qquad A_{23} = \begin{bmatrix} a'' & \sigma a'' \\ \tau\sigma a'' & \tau a'' \end{bmatrix} \qquad A_{12} = \begin{bmatrix} a & -\tau a \\ \sigma a & -\sigma\tau a \end{bmatrix}. \tag{10}$$

Since ${}^tA_{ii} = -A_{ii}$, we obtain

$$\begin{aligned} A_{11} &= \begin{bmatrix} 0 & b \\ \sigma b & 0 \end{bmatrix} && \text{with} \quad b = \tau b = -\sigma b \\ A_{22} &= \begin{bmatrix} 0 & b' \\ \tau b' & 0 \end{bmatrix} && \text{with} \quad b' = -\tau b' = \sigma b' \\ A_{33} &= \begin{bmatrix} 0 & b'' \\ \sigma b'' & 0 \end{bmatrix} && \text{with} \quad b'' = -\sigma b'' = \tau\sigma b''. \end{aligned} \tag{11}$$

Using table 1, we finally obtain

$$\begin{aligned}
A_{11} &= \begin{bmatrix} 0 & f_1 \\ -f_1 & 0 \end{bmatrix} \qquad & f_1 &= \alpha_3(v_3+v_4)+\alpha_5(v_5+v_6) \\
A_{22} &= \begin{bmatrix} 0 & f_2 \\ -f_2 & 0 \end{bmatrix} \qquad & f_2 &= \beta_1(v_1-v_2)+\beta_5(v_5-v_6) \\
A_{33} &= \begin{bmatrix} 0 & f_3 \\ -f_3 & 0 \end{bmatrix} \qquad & f_3 &= \gamma_1(v_1+v_2)+\gamma_3(v_3-v_4).
\end{aligned} \tag{12}$$

Proposition 6. *There is a system with Poisson matrix (8) and entries (10) and (12) that has the functions* $q_2 = v_3v_4$, $q_3 = v_5v_6$ *as Casimirs. Up to a change of basis, it is the* $SO(4)$ *case of Adler and Van Moerbeke.*

Proof. Since $J \cdot \operatorname{grad} q_2 = 0$, $J \cdot \operatorname{grad} q_3 = 0$, namely $J \cdot [0,0,v_4,v_3,0,0]^t = 0$, $J \cdot [0,0,0,0,v_6,v_5]^t = 0$. These impose conditions on A_{13}, A_{23}, A_{12}, A_{22} and A_{33}. Clearly, $f_2 = 0$ and $f_3 = 0$. Also

$$av_4 - \tau a v_3 = 0 \qquad a'v_6 - \tau a'v_5 = 0 \qquad -\sigma\tau a'v_6 + \sigma a'v_5 = 0$$

$$a''v_6 + \tau a''v_5 = 0 \qquad \tau\sigma a''v_6 + \tau a''v_5 = 0 \qquad a''v_4 + \tau\sigma a''v_3 = 0$$

$$\sigma a''v_4 + \tau a''v_3 = 0 \qquad av_4 - \tau a v_3 = 0 \Rightarrow a = \alpha_3'v_3$$

$$a'v_6 - \tau a'v_5 = 0 \quad \Rightarrow \quad a' = \beta_5'v_5 \qquad a''v_6 + \sigma a''v_5 = 0 \quad \Rightarrow \quad a'' = \gamma_5'v_5.$$

Thus

$$\begin{aligned}
A_{12} &= \begin{bmatrix} \alpha_3'v_3 & -\alpha_3v_4 \\ -\alpha_3'v_3 & \alpha_3'v_4 \end{bmatrix} \\
A_{13} &= \begin{bmatrix} \beta_5'v_5 & -\beta_5'v_6 \\ \beta_5'v_5 & -\beta_5'v_6 \end{bmatrix} \\
A_{23} &= \begin{bmatrix} \gamma_5'v_5 & -\gamma_5'v_6 \\ -\gamma_5'v_5 & \gamma_5'v_6 \end{bmatrix}
\end{aligned} \tag{13}$$

but the equations imply $A_{23} = 0$.

The $SO(4)$ system for the metric II is the system of differential equations [2, 9]:

$$\begin{aligned}
\dot\tau_1 &= \tau_2\tau_6 \qquad & \dot\tau_4 &= \tau_3\tau_5 \\
\dot\tau_2 &= \tfrac{1}{2}\tau_3(\tau_1+\tau_4) \qquad & \dot\tau_5 &= \tau_3\tau_4 \\
\dot\tau_3 &= \tfrac{1}{2}\tau_2(\tau_1+\tau_4) \qquad & \dot\tau_6 &= \tau_1\tau_2.
\end{aligned}$$

One can pick a Poisson matrix for this system

$$J_{SO(4)} = \begin{bmatrix}
0 & \tau_3 & \tau_2 & 0 & 0 & (2\tau_2-\tau_5) \\
-\tau_3 & 0 & 0 & 0 & 0 & 0 \\
-\tau_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \tau_5 \\
0 & 0 & 0 & 0 & 0 & \tau_4 \\
-(2\tau_2-\tau_5) & 0 & 0 & -\tau_5 & -\tau_4 & 0
\end{bmatrix}$$

with Poisson bracket

$$\{f, g\} = \left\langle \frac{\partial f}{\partial \tau}, J_{SO(4)} \cdot \frac{\partial g}{\partial \tau} \right\rangle.$$

By making the change of variables

$$v_1 = \tau_1 + \tau_6 \qquad v_2 = \tau_6 - \tau_1 \qquad v_3 = \tau_2 + \tau_3$$
$$v_4 = \tau_2 - \tau_3 \qquad v_5 = \tau_4 + \tau_5 \qquad v_6 = \tau_5 - \tau_4$$

the Poisson matrix takes the form

$$J_{SO(4)} = \begin{bmatrix} 0 & v_3 + v_4 - 1/2(v_5 + v_6) & v_3 & -v_4 & -v_5 & v_6 \\ -(v_3 + v_4) + 1/2(v_5 + v_6) & 0 & -v_3 & v_4 & -v_5 & v_6 \\ -v_3 & v_3 & 0 & 0 & 0 & 0 \\ v_4 & -v_4 & 0 & 0 & 0 & 0 \\ v_5 & v_5 & 0 & 0 & 0 & 0 \\ v_6 & -v_6 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} A & B & C \\ -{}^tB & 0 & 0 \\ -{}^tC & 0 & 0 \end{bmatrix}$$

and the invariants are $Q_1 = v_1 v_2,\ Q_2 = v_3 v_4,\ Q_3 = v_5 v_6;\ Q_4 = \frac{1}{2}(v_4 + v_3 - v_5 - v_6)^2 + \frac{1}{2}(v_3 - v_4 - v_1 - v_2)^2 - \frac{1}{4}(v_1 - v_2 - v_5 + v_6)^2$ with Q_2 and Q_3 as the Casimirs.

Another linear change of variables $w_i = f_i(v_1, \ldots, v_6)$ with matrix

$$\left(\frac{\partial w_i}{\partial v_j}\right) = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

brings the $J_{SO(4)}$ to the desired matrix. Indeed,

$$J_{SO(4)} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} A & B & C \\ -{}^tB & 0 & 0 \\ -{}^tC & 0 & 0 \end{bmatrix} \begin{bmatrix} {}^tB_{11} & {}^tB_{21} & {}^tB_{31} \\ {}^tB_{12} & {}^tB_{22} & {}^tB_{32} \\ {}^tB_{13} & {}^tB_{23} & {}^tB_{33} \end{bmatrix}$$
$$= \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$
$$\times \begin{bmatrix} A^tB_{11} + B^tB_{12} + C^tB_{13} & A^tB_{21} + B^tB_{22} + C^tB_{23} & A^tB_{31} + B^tB_{32} + C^tB_{33} \\ -{}^tB^tB_{11} & -{}^tB^tB_{21} & -{}^tB^tB_{31} \\ -{}^tC^tB_{11} & -{}^tC^tB_{21} & -{}^tCB_{31} \end{bmatrix}$$

pick $B_{21} = B_{31} = 0$, $B_{11} = I$, $B_{22} = \alpha_3' I$, $B_{23} = 0$, $B_{32} = 0$, $B_{33} = -\beta_5' I$.

The upper-left corner of the product matrix is $A' = A + B^t B_{12} - (B^t B_{12})^t + C^t B_{13} - (C^t B_{13})^t$, and if

$${}^tB_{12} = a \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \qquad {}^tB_{13} = b \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

for convenient scalars a and b, we obtain $A' = A_{11}$ as in (12). The other entries are treated similarly. □

8. Conclusions

As seen in this paper, the procedure of assigning the algebro-geometric data $(A_\alpha, \mathcal{D}_\alpha, \mathcal{L}_\alpha, G)$ (A_α an Abelian variety, $\mathcal{D}_\alpha$ a divisor on it, $\mathcal{L}_\alpha$ a line bundle on A_α and G a group (of translates) leaving $\mathcal{D}_\alpha$ and $\mathcal{L}_\alpha$ invariant) to an ACI system, can be accomplished in some cases in a successful way. There is a question of how unique a system is obtained from a given data. The question for the $SO(4)$ case would be settled if the 80 different configurations referred to in section 4 were shown to be isomorphic to the $SO(4)$ system. A more invariant way of looking at these problems is needed. If such a problem is carried out successfully one can ask whether it is possible to characterize such systems as the three-body periodic Toda lattice, Kowalevski's top or other ACI systems with two degrees of freedom. A more interesting question is whether these procedures allow one to find new 'mathematical' integrable systems. This would be a nice outcome for these techniques.

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